Rainbow connection of graphs with diameter 2*

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Abstract

A path in an edge-colored graph G, where adjacent edges may have the same color, is called a rainbow path if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of G is the minimum integer i for which there exists an i-edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. It is known that for a graph G with diameter 2, to determine $rc(G)$ is NP-hard. So, it is interesting to know the best upper bound of $rc(G)$ for such a graph G. In this paper, we show that $rc(G) \leq 5$ if G is a bridgeless graph with diameter 2, and that $rc(G) \leq k+2$ if G is a connected graph of diameter 2 with k bridges, where $k \geq 1$.

Keywords: Edge-coloring, Rainbow path, Rainbow connection number, Diameter

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to book [1] for graph theoretical notation and terminology not described here. A path in an edge-colored graph G, where adjacent edges may have the same color, is called a *rainbow path* if no two edges of the path are colored the same. An edge-coloring of graph G is a *rainbow edge-coloring* if every two distinct vertices of graph G are connected by a rainbow path. The *rainbow connection number* $rc(G)$ of G is the minimum integer i for which there exists an i-edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. It is easy to see that $diam(G) \leq rc(G)$ for any connected graph G, where $diam(G)$ is the diameter of G.

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The rainbow connection number was introduced by Chartrand et al. in [5]. It is of great use in transferring information of high security in multicomputer networks. We refer the readers to [3, 6] for details.

Chartrand et al. [5] considered the rainbow connection number of several graph classes and showed the following proposition and theorem.

Proposition 1. *[5] Let G be a nontrivial connected graph of size m. Then*

(i) $src(G) = 1$ if and only if G is a complete graph;

(ii) $rc(G) = m$ if and only if G is a tree;

(iii) $rc(C_n) = \lceil n/2 \rceil$ for each integer $n \geq 4$, where C_n is a cycle with size n.

Theorem 1. *[5] For integers s and t with $2 \leq s \leq t$,*

$$rc(K_{s,t}) = \min\{\lceil \sqrt[s]{t} \rceil, 4\},$$

where $K_{s,t}$ is the complete bipartite graph with bipartition X and Y, such that $|X| = s$ and $|Y| = t$.

Krivelevich and Yuster [7] investigated the relation between the rainbow connection number and the minimum degree of a graph, and showed the following theorem.

Theorem 2. *[7] A connected graph G with n vertices and minimum degree δ has $rc(G) < \frac{20n}{\delta}$.*

In fact, Krivelevich and Yuster [7] made the following conjecture.

Conjecture 1. *[7] If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{3n}{4}$.*

Schiermeyer showed that the above conjecture is true by the following theorem, and that the following bound is almost best possible since there exist 3-regular connected graphs with $rc(G) = \frac{3n-10}{4}$.

Theorem 3. *[8] If G is a connected graph with n vertices and $\delta(G) \geq 3$ then,*

$$rc(G) \leq \frac{3n-1}{4}.$$

Chandran et al. studied the rainbow connection number of a graph by means of connected dominating sets. A dominating set D in a graph G is called a *two-way dominating set* if every pendant vertex of G is included in D. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*.

Theorem 4. *[4] If D is a connected two-way dominating set of a graph G, then*

$$rc(G) \leq rc(G[D]) + 3.$$

Let G be a graph. The *eccentricity* of a vertex u, written as $\epsilon_G(u)$, is defined as $\max\{d_G(u,v) \mid v \in V(G)\}$. The *radius* of a graph, written as $rad(G)$, is defined as $\min\{\epsilon_G(u) \mid u \in V(G)\}$. A vertex u is called a *center* of a graph G if $\epsilon_G(u) = rad(G)$.

Basavaraju et al. evaluated the rainbow connection number of a graph by its radius and *chordality (size of a largest induced cycle)*, and presented the following theorem.

Theorem 5. *[2] For every bridgeless graph G,*

$$rc(G) \leq rad(G)\zeta(G),$$

where $\zeta(G)$ is the size of a largest induced cycle of the graph G.

They also showed that the above result is best possible by constructing a kind of tight examples.

Chakraborty et al. investigated the hardness and algorithms for the rainbow connection number, and showed the following theorem.

Theorem 6. *[3] Given a graph G, deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.*

It is well-known that almost all graphs have diameter 2. So, it is interesting to know the best upper bound of $rc(G)$ for a graph G with diameter 2. Clearly, the best lower bound of $rc(G)$ for such a graph G is 2. In this paper, we give the upper bound of the rainbow connection number of a graph with diameter 2. We show that if G is a bridgeless graph with diameter 2, then $rc(G) \leq 5$, and that $rc(G) \leq k+2$ if G is a connected graph of diameter 2 with k bridges, where $k \geq 1$.

The end of each proof is marked by a $\Box$. For a proof consisting of several claims, the end of the proof of each claim is marked by a $\triangle$.

2 Main results

We need some notations and terminology first. Let G be a graph. The *k-step open neighbourhood* of a vertex u in G is defined by $N_G^k(u) = \{v \in V(G) \mid d_G(u,v) = k\}$ for $0 \leq k \leq\ diam(G)$. We write $N_G(u)$ for $N_G^1(u)$ simply. Let X be a subset of $V(G)$, and denote by $N_G^k(X)$ the set $\{u \mid d_G(u,X) = k, u \in V(G)\}$, where $d_G(u,X) = \min\{d_G(u,x) \mid x \in X\}$. For any two subsets X, Y of $V(G)$, $E_G[X,Y]$ denotes the set $\{xy \mid x \in X, y \in Y, xy \in E(G)\}$. Let c be a rainbow edge-coloring of G. If an edge e is colored by i, we say that e is an *i-color edge*. Let P be a rainbow path. If $c(e) \in \{i_1, i_2, \ldots, i_r\}$ for any $e \in E(P)$, then P is called an *$\{i_1, i_2, \ldots, i_r\}$-rainbow path*. Let $X_1, X_2, \ldots X_k$ be disjoint vertex subsets of G. Notation $X_1 - X_2 - \cdots - X_k$ means that there exists some desired rainbow path $P = (x_1, x_2, \ldots, x_k)$, where $x_i \in X_i,\ i = 1, 2, \ldots, k$.

Theorem 7. *Let G be a connected graph of diameter 2 with $k \geq 1$ bridges. Then $rc(G) \leq k+2$.*

Proof. G must have a cut vertex, say v, since G has bridges. Furthermore, v must be the only cut vertex of G, and the common neighbor of all other vertices due to $diam(G) = 2$. Let $G_1, G_2, \ldots, G_r$ be the components of $G - v$. Without loss of generality, assume that $G_1, G_2, \ldots, G_k$ are the all trivial components of $G - v$. We consider the following two cases to complete this proof.

Case 1. $k = r$.

In this case, we provide each bridge with a distinct color from $\{1, 2, \ldots, k\}$. It is easy to see that this is a rainbow edge-coloring. Thus $rc(G) \leq k \leq k+2$.

Case 2. $k < r$.

In this case, first provide each bridge with a distinct color, and denote by c_1 this edge-coloring. Next color the other edges as follows. Let F be a spanning forest of the disjoint union $G_{k+1} + G_{k+2} + \cdots + G_r$ of $G_{k+1}, G_{k+2}, \ldots, G_r$, and let X and Y be any one of the bipartition defined by this forest F. We provide a 3-edge-coloring $c_2 : E(G_{k+1} + G_{k+2} + \cdots + G_r) \to \{1, k+1, k+2\}$ of G defined by

$$c_2(e) = \begin{cases} k+1, & if\ e \in E[v, X]; \\ k+2, & if\ e \in E[v, Y]; \\ 1, & otherwise. \end{cases}$$

We show that the edge-coloring $c_1 \cup c_2$ is a rainbow edge-coloring of G in this case. Pick any two distinct vertices u and w in $V(G)$. If one of u and w is v, then $u - w$ is a rainbow path. If at least one of u and w is a trivial component of $G - v$, then u, v, w is a rainbow path connecting u and w. Thus we suppose $u, w \in X \cup Y$. If $u \in X$ and $w \in Y$, or $w \in X$ and $u \in Y$, then u, v, w is a rainbow path connecting u and w. If $u, w \in X$, or $u, w \in Y$, without loss of generality, assume $u, w \in X$. Pick $z \in Y$ such that $uz \in E[F]$. Thus u, z, v, w is a rainbow path connecting u and w. So $rc(G) \leq k+2$.

By this all possibilities have been exhausted and the proof is thus complete. □

Tight examples: The upper bound of Theorem 7 is tight. The graph $(kK_1 \cup rK_2) \vee v$ has a rainbow connection number achieving this upper bound, where $k \geq 1, r \geq 2$.

Proposition 2. *Let G be a bridgeless graph with order n and diameter 2. Then G is either 2-connected, or G has only one cut vertex v. Furthermore, v is the center of G with radius 1.*

Proof. Let G be a bridgeless graph with diameter 2. Suppose that G is not 2-connected, that is, G has a cut vertex. Since $diam(G) = 2$, G has only one cut vertex, say v. Let $G_1, G_2, \ldots, G_k$ be the components of $G - v$ where $k \geq 2$. If some vertex, without loss of

generality, say $u \in V(G_1)$, is not adjacent to v, then $d_G(u, w) \geq 3$ for any $w \in V(G_2)$. This conflicts with the fact that $diam(G) = 2$. So v is the center of G with radius 1. $\square$

Lemma 1. *Let G be a bridgeless graph with diameter 2. If G has a cut vertex, then $rc(G) \leq 3$.*

Remark 1. *This lemma can be proved by a similar argument for Theorem 7. It can also be derived from Theorem 5.*

Lemma 2. *Let G be a 2-connected graph with diameter 2. Then $rc(G) \leq 5$.*

Proof. Pick a vertex v in $V(G)$ arbitrarily. Let

$$B = \{u \in N_G^2(v) \,|\, \textit{there exists a vertex } w \in N_G^2(v) \textit{ such that } uw \in E(G)\}.$$

We consider the following two cases distinguishing either $B \neq \emptyset$ or $B = \emptyset$.

Case 1. $B \neq \emptyset$.

In this case, the subgraph $G[B]$ induced by B has no isolated vertices. Thus there exists a spanning forest F in $G[B]$, which also has no isolated vertices. Furthermore, let B_1 and B_2 be any one of the bipartition defined by this forest F. Now divide $N_G(v)$ as follows.

Set $X, Y = \emptyset$. For any $u \in N_G(v)$, if $u \in N_G(B_1)$, then put u into X. If $u \in N_G(B_2)$, then put u into Y. If $u \in N_G(B_1)$ and $u \in N_G(B_2)$, then put u into X. By the above argument, we know that for any $x \in X$ ($y \in Y$), there exists a vertex $y \in Y$ ($x \in X$) such that x and y are connected by a path P with length 3 satisfying $(V(P) \setminus \{x, y\}) \subseteq B$.

We have the following claim for any $u \in N_G(v) \setminus (X \cup Y)$.

Claim 1. *Let $u \in N_G(v) \setminus (X \cup Y)$. Then either u has a neighbor $w \in X$, or u has a neighbor $w \in Y$.*

Proof of Claim 1. Let $u \in N_G(v) \setminus (X \cup Y)$. Pick $z \in B_1$, then u and z are nonadjacent since $u \notin X \cup Y$. Moreover, $diam(G) = 2$, so u and z have a common neighbor w. We say that $w \notin N_G^2(v)$, otherwise, $w \in B$ and $u \in X \cup Y$, which contradicts the fact that $u \notin X \cup Y$. Moreover, we say that $w \notin N_G(v) \setminus (X \cup Y)$ by a similar argument. Thus w must be contained in $X \cup Y$. $\triangle$

By the above claim, for any $u \in N_G(v) \setminus (X \cup Y)$, either we can put u into X such that $u \in N_G(Y)$, or we can put u into Y such that $u \in N_G(X)$. Now X and Y form a partition of $N_G(v)$.

For any $u \in N_G^2(v) \setminus B$, let

$$A = \{u \in N_G^2(v) \,|\, u \in N_G(X) \cap N_G(Y)\};$$

$$D_1 = \{u \in N_G^2(v) \,|\, u \in N_G(X) \setminus N_G(Y);$$

$$D_2 = \{u \in N_G^2(v) \mid u \in N_G(Y) \setminus N_G(X)\}.$$

We say that at least one of D_1 and D_2 is empty. Otherwise, there exist $u \in D_1$ and $v \in D_2$ such that $d_G(u, v) \geq 3$, which contradicts the fact that $diam(G) = 2$. Without loss of generality, assume $D_2 = \emptyset$.

First, we provide a 5-edge-coloring $c : E(G) \setminus E_G[D_1, X] \to \{1, 2, \ldots, 5\}$ defined by

$$c(e) = \begin{cases} 1, & if\ e \in E_G[v, X]; \\ 2, & if\ e \in E_G[v, Y]; \\ 3, & if\ e \in E_G[X, Y] \cup E_G[Y, A] \cup E_G[B_1, B_2]; \\ 4, & if\ e \in E_G[X, A] \cup E_G[X, B_1]; \\ 5, & if\ e \in E_G[Y, B_2],\ or\ otherwise. \end{cases}$$

Next, we color the edges of $E_G[X, D_1]$ as follows. For any vertex $u \in D_1$, color one edge incident with u by 5 (solid lines), the other edges incident with u are colored by 4 (dotted lines). See Figure 1.

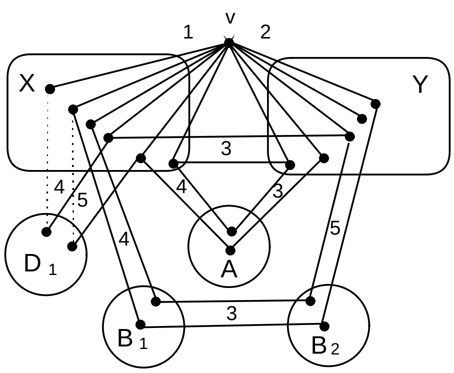

Figure 1.

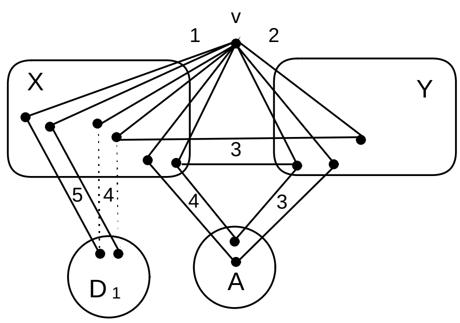

Figure 2

We have the following claim for the above coloring.

Claim 2. (i) *For any vertex $x \in X$, there exists a vertex $y \in Y$ such that x and y are connected by a $\{3, 4, 5\}$-rainbow path in $G - v$.*

(ii) *For any vertex $y \in Y$, there exists a vertex $x \in X$ such that x and y are connected by a $\{3, 4, 5\}$-rainbow path in $G - v$.*

(iii) *For any $u, u' \in D_1$, there exists a rainbow path connecting u and u'.*

(iv) *For any $u \in D_1$ and $u' \in X$, there exists a rainbow path connecting u and u'.*

Proof of Claim 2. First, we show that (i) and (ii) hold. We only prove part (i), since part (ii) can be proved by a similar argument. By the procedure of constructing X and Y, we know that for any $x \in X$, either there exists a vertex $y \in Y$ such that $xy \in E(G)$, or there exists a vertex $y \in Y$ such that x and y are connected by a path P with length 3 satisfying $(V(P) \setminus \{x, y\}) \subseteq B$. Clearly, this path is a $\{3, 4, 5\}$-rainbow path.

Next, we show that (iii) holds. Let $u, u' \in D_1$. For any $y \in Y$, since $diam(G) = 2$, we have that u and y have a common adjacency vertex $w \in X$. Furthermore, without loss of generality, assume that uw has color 5. Then $u - w - y - v - w' - u'$ is a rainbow path connecting u and u', where u' is adjacent to w' by a 4-color edge $u'w'$.

Finally, we show that (iv) holds. For any $y \in Y$, since $diam(G) = 2$, we have that u and y have a common adjacency vertex $w \in X$. Thus $u - w - y - v - u'$ is a rainbow path connecting u and u'. $\triangle$

It is easy to see that the above edge-coloring is rainbow in this case from Figure 1 and Table 1.

	v	X	Y	A	B_1	B_2	D_1
v	—	$v - X$	$v - Y$	$v - X - A$	$v - X - B_1$	$v - X - B_1 - B_2$	$v - X - D_1$
X	—	Claim 2 and $Y - v - X$	$X - v - Y$	$X - v - Y - A$	$X - v - Y - B_2 - B_1$	$X - v - Y - B_2$	Claim 2
Y	—	—	Claim 2 and $X - v - Y$	$Y - v - X - A$	$Y - v - X - B_1$	$Y - v - X - B_1 - B_2$	$Y - v - X - D_1$
A	—	—	—	$A - X - v - Y - A$	$A - Y - v - X - B_1$	$A - X - v - Y - B_2$	$A - Y - v - X - D_1$
B_1	—	—	—	—	$B_1 - X - v - Y - B_2 - B_1$	$B_1 - X - v - Y - B_2$	$B_1 - B_2 - Y - v - X - D_1$
B_2	—	—	—	—	—	$B_2 - B_1 - X - v - Y - B_2$	$B_2 - Y - v - X - D_1$
D_1	—	—	—	—	—	—	Claim 2

Table 1. The rainbow paths in G

Case 2. $B = \emptyset$.

In this case, clearly, $N_G(u) \subseteq N_G(v)$ for any $u \in N_G^2(v)$. To show a rainbow coloring of G, we need to construct a new graph H. The vertex set of H is $N_G(v)$, and the edge set is $\{xy \,|\, x, y \in N_G(v),\ x$ and y are connected by a path P with length at most 2 in $G - v$, and $V(P) \cap N_G(v) = \{x, y\}$.

Claim 3. *The graph H is connected.*

Proof of Claim 3. Let x and y be any two distinct vertices of H. Since G is 2-connected, x and y are connected by a path in $G - v$. Assume that $P = (x = v_0, v_1, \ldots, v_k = y)$ is a shortest path between x and y in $G - v$.

If $k = 1$, then by the definition of H, x and y are adjacent in H. Otherwise, $k \geq 2$. Since $diam(G) = 2$, v_i is adjacent to v, or v_i and v have a common neighbor u_i if $d_G(v, v_i) = 2$. For any integer $0 \leq i \leq k - 1$, if $d_G(v, v_i) = 1$ and $d_G(v, v_{i+1}) = 1$, then v_i and v_{i+1} are contained in $V(H)$, and adjacent in H. If $d_G(v, v_i) = 1$ and $d_G(v, v_{i+1}) = 2$, then v_i and u_{i+1} are contained in $V(H)$, and adjacent in H. If $d_G(v, v_i) = 2$ and $d_G(v, v_{i+1}) = 1$, then u_i and v_{i+1} are contained in $V(H)$, and adjacent in H. If $d_G(v, v_i) = 2$ and $d_G(v, v_{i+1}) = 2$, then u_i and u_{i+1} should be contained in B, which contradicts the fact that $B = \emptyset$. Thus, there exists a path between x and y in H. The proof of Claim 3 is complete. $\triangle$

Let T be a spanning tree of H, and let X and Y be the bipartition defined by this tree. Now divide $N_G^2(v)$ as follows: for any $u \in N_G^2(v)$,

$$let\ A = \{u \in N_G^2(v) \mid u \in N_G(X) \cap N_G(Y)\};$$

and for any $u \in N_G^2(v) \setminus A$,

$$let\ D_1 = \{u \in N_G^2(v) \mid u \in N_G(X) \setminus N_G(Y)\},$$

$$D_2 = \{u \in N_G^2(v) \mid u \in N_G(Y) \setminus N_G(X)\}.$$

We say that at least one of D_1 and D_2 is empty. Otherwise, there exist $u \in D_1$ and $v \in D_2$ such that $d_G(u, v) \geq 3$, which contradicts the fact that $diam(G) = 2$. Without loss of generality, assume $D_2 = \emptyset$. Then A and D_1 form a partition of $N_G^2(v)$ (see Figure 2).

First, we provide a 4-edge-coloring $c : E(G) \setminus E_G[D_1, X] \to \{1, 2, \ldots, 4\}$ defined by

$$c(e) = \begin{cases} 1, & if\ e \in E_G[v, X]; \\ 2, & if\ e \in E_G[v, Y]; \\ 3, & if\ e \in E_G[X, Y] \cup E_G[Y, A]; \\ 4, & if\ e \in E_G[X, A],\ or\ otherwise. \end{cases}$$

Next, we color the edges of $E_G[X, D_1]$ as follows. For any vertex $u \in D_1$, color one edge incident with u by 5 (solid lines), the other edges incident with u are colored by 4 (dotted lines). See Figure 2.

Now, we show that the above edge-coloring is a rainbow in this case from Figure 2 and Table 2.

	v	X	Y	A	D_1
v	—	$v - X$	$v - Y$	$v - X - A$	$v - X - D_1$
X	—	Claim 2 and $Y - v - X$	$X - v - Y$	$X - v - Y - A$	Claim 2
Y	—	—	Claim 2 and $X - v - Y$	$Y - v - X - A$	$Y - v - X - D_1$
A	—	—	—	$A - X - v - Y - A$	$A - Y - v - X - D_1$
D_1	—	—	—	—	$D_1 - A - Y - v - X - D_1$

Table 2. The rainbow paths in G

By this both possibilities have been exhausted and the proof is thus complete. □

Combining Proposition 2 with Lemmas 1 and 2, we have the following theorem.

Theorem 8. *Let G be a bridgeless graph with diameter 2. Then $rc(G) \leq 5$.*

A simple graph G which is neither empty nor complete is said to be *strongly regular* with parameters (n, k, λ, μ), denoted by $SRG(n, k, \lambda, \mu)$, if (i) $V(G) = n$; (ii) G is k-regular; (iii) any two adjacent vertices of G have λ common neighbors; (iv) any two nonadjacent vertices of G have μ common neighbors. It is well known that a strongly regular with parameters (n, k, λ, μ) is connected if and only if $\mu \geq 1$.

Corollary 1. *If G is a strongly regular graph, other than a star, with $\mu \geq 1$, then $rc(G) \leq 5$.*

Proof. If $\mu \geq 2$, then G is 2-connected. Thus $rc(G) \leq 5$ by Theorem 8. If $\mu = 1$ and $\lambda \geq 1$, then G is bridgeless. Thus $rc(G) \leq 5$ by Theorem 8. Thus, the left case is that $\mu = 1$ and $\lambda = 0$.

First, suppose that G is a tree. Then $G \cong K_2$ since G is regular. But this contradicts the fact that G is a strongly regular graph.

Next, suppose that G is not a tree. We claim that all the induced cycles of G have length 5. If G has an induced cycle with length 3, then there exist two adjacent vertices u and v in C such that $|N_G(u) \cap N_G(v)| \geq 1$, which conflicts with $\lambda = 0$. If G has an induced cycle with length 4, then there exist two nonadjacent vertices u and v in C such that $|N_G(u) \cap N_G(v)| \geq 2$, which conflicts with $\mu = 1$. Otherwise, G has an induced cycle C with length at least 6. Then there exist two nonadjacent vertices u and v in C such that $|N_G(u) \cap N_G(v)| = 0$, which conflicts with $\mu = 1$.

We say that G is bridgeless. By contradiction, let $e = uv$ be a bridge. Then there exist two components, say G_1 and G_2, in $G - v$. Since G is not a tree, there exists a cycle C contained in G_1 (or G_2). Without loss of generality, assume that $u \in V(G_1)$ and $C \subseteq G_1$. Pick $w \in C$ such that $u \notin N_G(w)$ (There exists such a vertex, since all the induced cycles of G have length 5). Then v and w are nonadjacent, and $N_G(v) \cap N_G(w) = \emptyset$, which conflicts with $\mu = 1$. Thus G is bridgeless. Therefore $rc(G) \leq 5$ by Theorem 8. $\square$

Remark. From [4] we know that the complete bipartite graph $K_{2,n}$ has a diameter 2, and its rainbow connection number is 4 for $n \geq 10$. However, we failed to find an example for which the rainbow connection number reaches 5.

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